Automorphisms of Triangular Matrices Over Graphs

George Phillip Barker
Department of Mathematics
University of Missouri — Kansas City
5100 Rockhill Road
Kansas City, Missouri 64110-2499

Submitted by Hans Schneider

1. REVIEW AND NOTATION

We shall continue the study of the factorization of automorphisms of upper triangular matrices with elements from an algebra $\mathscr{A}$ over a commutative ring R. All rings and algebras are assumed to have a unity, denoted by 1. In particular we obtain a common generalization of the principal results of [1] and [3].

Let T be the graph whose vertex set is $\{1,\dots,n\}$ and whose edge set is all pairs (i,j) with $i \leqslant j$. If Γ is a subgraph of T with edge set $\{1,\dots,\}$, then the *complement* $\mathbf{H}$ of Γ is the graph on $\{1,\dots,n\}$ whose edge set is the complement of that of Γ in $\mathbf{T}$. We say that Γ is connected iff for any $i \leqslant j$ there is a sequence $i = j_1, j_2, \dots, j_p = j$ for which either (j_q, j_{q+1}) or (j_{q+1}, j_q) is an edge of Γ. Note that Γ is a directed graph. Then Γ is connected iff the underlying nondirected graph is connected in the usual sense. If σ is any permutation on $\{1,\dots,n\}$, we say that Γ is *invariant* under σ iff (i,j) an edge of Γ implies that $(\sigma(i),\sigma(j))$ is also an edge of Γ. Next, Γ is called *transitive* iff (i,j) and (j,k) edges of Γ imply that (i,k) is an edge of Γ. Finally, we say that Γ is *interpolatory* or satisfies the *interpolation property* iff whenever (i,j) and (i,k) are edges of Γ and $j < k$, then so is (j,k).

Now let R be a commutative ring and $\mathscr{A}$ an algebra over R (both with unity). Let Γ and $\mathbf{H}$ be complementary subgraphs of $\mathbf{T}$. Denote by $\mathscr{T}_n$ [or $\mathscr{T}_n(\Gamma)$ if necessary] the $\mathscr{A}$-module (and R-algebra) of all upper triangular matrices with entries from $\mathscr{A}$ generated by all E_{ij} with (i,j) an edge of Γ

655 Avenue of the Americas, New York, NY 10010 0024-3795/92/$3.50

[where E_{ij} is the standard matrix unit which has a 1 in position (i, j) and zeros elsewhere]. Note that by our definition if Γ is connected, transitive, and interpolatory, then (i, i) is an edge of Γ, so that $E_{ii} \in \mathscr{T}_n$ for $i = 1, \ldots, n$. For the motivation of these choices and the relations between graphs and algebras see [1], especially the lemma on p. 210 in connection with the next definition.

DEFINITION. Let Γ be a subgraph of $\mathscr{T}$ which contains all loops (i, i).

(a) Denote by Σ the set of all permutations σ of $\{1, \ldots, n\}$ such that Γ is invariant under σ.

(b) Denote by Π the set of all mappings

$$\pi(T) = P^{-1}TP,$$

where $T \in \mathscr{T}_n$, and P is the permutation matrix determined by π.

REMARK. Since Γ is invariant under π, it follows that Π is a subgroup of the group $\mathrm{Aut}(\mathscr{T}_n)$ of R-automorphisms of the R-algebra $\mathscr{T}_n$.

2. STATEMENT OF RESULTS

THEOREM. *Let $\mathscr{A}$ be an R-algebra with unity such that every nontrivial R-algebra endomorphism of $\mathscr{A}$ is an automorphism. Let Γ be a transitive connected interpolatory subgraph of $\mathscr{T}$, and let $\mathscr{T}_n$ denote the algebra of $n \times n$ upper triangular matrices with entries from $\mathscr{A}$ generated over Γ. If Θ is an R-automorphism of $\mathscr{T}_n$, then Θ can be factored as*

$$\Theta = \Phi \circ \Psi \circ \Omega,$$

where Φ is inner, $\Omega \in \Pi$, and Ψ is engendered by an R-automorphism ψ of $\mathscr{A}$ according to

$$\Psi([t_{ij}]) = [\psi(t_{ij})]. \qquad (*)$$

COROLLARY 1 [3]. *If $\mathcal{T}(\mathcal{A})$ is the R-algebra of all $n \times n$ upper triangular matrices over $\mathcal{A}$, then every R-automorphism Θ of $\mathcal{T}(\mathcal{A})$ can be factored as*

$$\Theta = \Phi \circ \Psi,$$

where Φ and Ψ are as in the proof.

Proof. In this case $\Gamma = T$. The only permutation under which T is invariant is the identity, so Ω is the identity. ■

COROLLARY [1]. *Let R be an integral domain, and let Γ be a transitive connected interpolatory subgraph of $\mathbf{T}$. Let $\mathcal{T}$ be the algebra of $n \times n$ upper triangular matrices with entries from R generated over Γ. Then every R-automorphism Θ of $\mathcal{T}$ can be factored as*

$$\Theta = \Phi \circ \Omega,$$

where Φ and Ω are as in the theorem.

Proof. If R is an integral domain and $\mathcal{A} = R$ in the statement of the theorem, then the only nontrivial R-endomorphism of R is the identity, so that Ψ is the identity. ■

REMARKS. Corollary 2 is Theorem 2 of [1] with the additional hypotheses that Γ is connected and interpolatory. However, these hypotheses seem to be needed (see the examples after the proof), and without them there is a gap in the proof of Theorem 2. The gap occurs at the bottom of p. 212 in our paper [2]. The present proof is a modification of those in [1] and [3]. It is worth noting that if $\Gamma = \mathbf{T}$ and $\mathcal{A} = R$, a commutative ring, then in [4] Kezlan has shown that every R-automorphism of $\mathcal{T}_n$ is inner.

3. PROOF OF THE THEOREM

We shall induct on n. First note that for any Ψ and Ω of the given forms we have

$$\Psi \circ \Omega = \Omega \circ \Psi. \qquad (C_1)$$

Further, since the group of inner automorphisms is normal in the group of all automorphisms, then for any Ψ and Ω and any inner automorphism Φ there are inner automorphisms Φ_1 and Φ_2 such that

$$\Phi \circ \Psi = \Psi \circ \Phi_1, \qquad \Phi \circ \Omega = \Omega \circ \Phi_2. \tag{C$_2$}$$

Subsequently we shall refer to (C_1) and (C_2) as the commutation relations.

Now for $n = 1$ the conclusion clearly holds, so suppose it is true for algebras of matrices of order $k \leqslant n-1$, and let $k = n$. Let Θ be an automorphism of $\mathscr{T}_n$. For any $a \in \mathscr{A}$ define ψ by

$$\psi(a) = \left[\Theta^{-1}(aI)\right]_{11},$$

where $[T]_{ij}$ denotes the (i, j) element of $T \in \mathscr{T}_n$. Clearly, ψ is R-linear, so we show that it is multiplicative. Let $a, b \in \mathscr{A}$. We have

$$\begin{aligned}\psi(ab) &= \left[\Theta^{-1}(abI)\right]_{11} = \left[\Theta^{-1}(aI)\Theta^{-1}(bI)\right]_{11}\\ &= \left[\Theta^{-1}(aI)\right]_{11}\left[\Theta^{-1}(bI)\right]_{11} = \psi(a)\psi(b),\end{aligned}$$

where we use the observation that for upper triangular matrices S and T, necessarily $[ST]_{11} = [S]_{11}[T]_{11}$. Thus ψ is an R-endomorphism of $\mathscr{A}$. Further,

$$\psi(1) = \left[\Theta^{-1}(1I)\right]_{11} = [I]_{11} = 1,$$

whence ψ is nontrivial and so by hypothesis is an R-automorphism. We use $(*)$ to define Ψ_1 in terms of ψ. Note that for any $T = [t_{ij}] \in \mathscr{T}_n$, $\Psi_1^{-1}(T) = [\psi^{-1}(t_{ij})]$, so that Ψ_1^{-1} is also defined by $(*)$. Put

$$\Theta_1 = \Theta \circ \Psi_1.$$

For any $a \in \mathscr{A}$ we obtain

$$\begin{aligned}\Theta_1(aE_{11}) &= \Theta \circ \Psi_1(aE_{11}) = \Theta\left[\psi(a)E_{11}\right]\\ &= \Theta\left(\left[\Theta^{-1}(aI)\right]_{11}E_{11}\right) = \Theta\left(\Theta^{-1}(aI)E_{11}\right) = (aI)\Theta(E_{11})\\ &= a\Theta(\psi(1)E_{11}) = a\Theta \circ \Psi_1(E_{11}) = a\Theta_1(E_{11}).\end{aligned}$$

We may summarize this calculation by saying that for all $a \in \mathscr{A}$

$$\Theta_1(aE_{11}) = a\Theta_1(E_{11}). \qquad (**)$$

Now since $\mathscr{T}_n E_{11} = \mathscr{A}E_{11}$, we may use $(**)$ to infer that

$$\mathscr{T}_n\Theta_1(E_{11}) = \mathscr{A}\Theta_1(E_{11}). \qquad (**')$$

At this point we need to examine the diagonals of each $\Theta_1(E_{jj})$. For each j, $1 \leqslant j \leqslant n$, define a set $\{\xi_{ij}\}_{i=1}^n$ of R-endomorphisms from $\mathscr{A}$ into $\mathscr{A}$ by

$$\Theta_1(aE_{jj}) = \begin{bmatrix} \xi_{1j}(a) & & * \\ & \ddots & \\ 0 & & \xi_{nj}(a) \end{bmatrix}$$

for $a \in \mathscr{A}$. Since each ξ_{ij} is an R-algebra endomorphism of $\mathscr{A}$, then by hypothesis each must be either trivial or an automorphism. Let $e_{ij} = \xi_{ij}(1)$, so that e_{ij} is either 0 or 1. Since $\sum_{j=1}^n \Theta_1(E_{jj}) = I$, we must have $\sum_{j=1}^n e_{ij} = 1$ for $1 \leqslant i \leqslant n$. For a given i the e_{ij}, $1 \leqslant j \leqslant n$, are mutually orthogonal idempotents. Thus for a given i, if $e_{ik} = 1$, then $e_{il} = 0$ for $l \neq k$. Thus $\{e_{ij}\}_{j=1}^n$ consists of precisely one 1 and the rest 0. Moreover, since $\Theta_1(E_{jj})$ is not nilpotent, it must have at least one 1 on its diagonal. It follows that each $\Theta_1(E_{jj})$ has one and only one diagonal entry equal to 1, while the rest are 0, with the 1's in different positions for different j's. In short, there is a permutation σ of $\{1, \ldots, n\}$ such that

$$\Theta_1(E_{jj}) = E_{\sigma(j)\sigma(j)} + S_j$$

where S_j is strictly upper triangular. Thus from $(**')$ we have

$$E_{\sigma(1)\sigma(1)}\Theta_1(E_{11}) = a\Theta_1(E_{11}),$$

or

$$E_{\sigma(1)\sigma(1)} + E_{\sigma(1)\sigma(1)}S_j = aE_{\sigma(1)\sigma(1)} + aS_1,$$

whence $a = 1$ and all the nonzero entries of $\Theta_1(E_{11})$ occur in row $\sigma(1)$.

We have two cases.

Case 1: *σ is the identity*. In particular $\sigma(1)=1$ and we have

$$\Theta_1(E_{11})=\begin{bmatrix} 1 & c_2 & \cdots & c_n \\ & & 0 & \end{bmatrix}.$$

Let

$$T=\begin{bmatrix} 1 & c_2 & \cdots & c_n \\ 0 & & I_{n-1} & \\ 0 & & & \end{bmatrix}.$$

Then $T=\Theta_1(E_{11})+I_n-E_{11}\in\mathscr{T}_n$, so $T^{-1}\in\mathscr{T}_n$ as well, whence $\Phi_1(\mathscr{A})=T^{-1}\mathscr{A}T$ is an inner automorphism of $\mathscr{T}_n$. Note that $\Theta_1(E_{11})=T^{-1}E_{11}T$, and define Θ_2 by $\Theta_1=\Phi_1\circ\Theta_2$. We wish to verify that $\Theta_2(aE_{11})=aE_{11}$. Note that by $(**)$

$$\Theta_1(aE_{11})=a\Theta_1(E_{11})=aE_{11}T.$$

Therefore

$$\Theta_2(aE_{11})=\Phi_1^{-1}(\Theta_1(aE_{11}))=T(aE_{11}T)T^{-1}=TaE_{11}=aE_{11}.$$

Also we have $\Theta=\Theta_1\circ\Psi_1^{-1}=\Phi_1\circ\Theta_2\circ\Psi_1^{-1}$, where Φ_1 and Ψ_1^{-1} are of the prescribed form. Thus we may work with Θ_2.

To deal with Θ_2 we let

$$\mathscr{S}=\{0\oplus U : 0 \text{ is } 1\times 1 \text{ and } U\in\mathscr{T}_{n-1}\},$$

where $\mathscr{T}_{n-1}$ consists of all principal submatrices of elements of $\mathscr{T}_n$ on rows and columns $2,\dots,n$. If $T\in\mathscr{T}_n$, then $T-E_{11}T\in\mathscr{S}$, so we see that $\mathscr{S}$ is in fact a subalgebra of $\mathscr{T}_n$. If Γ^1 is the graph on $2,\dots,n$ whose edges are the edges (i,j) of Γ with $2\leqslant i$, then Γ^1 is likewise transitive and $\mathscr{T}_{n-1}$ is the algebra generated over Γ^1. Observe that $T\in\mathscr{S}$ iff $E_{11}T=0=TE_{11}$, and this holds iff $E_{11}\Theta_2(T)=0=\Theta_2(T)E_{11}$, as $\Theta_2(E_{11})=E_{11}$. Hence $T\in\mathscr{S}$ iff $\Theta_2(T)\in\mathscr{S}$. Thus Θ_2 restricted to $\mathscr{S}$ can be regarded as an R-automorphism of $\mathscr{T}_{n-1}$. However, Γ^1 need not be connected. Let $\Gamma^1=\Delta_1\oplus\cdots\oplus\Delta_p$, where the Δ_j are the components of Γ^1. Let $\mathscr{S}_i$ be the R-algebra and $\mathscr{A}$-module generated by all E_{hk} with $(h,k)\in\Delta_i$, and let $\mathscr{T}_{i\,n-1}$ be the corresponding subalgebra of $\mathscr{T}_{n-1}$. Then $\mathscr{S}=\mathscr{S}_1\oplus\cdots\oplus\mathscr{S}_p$ (where the sum is an algebra and a module direct sum). We claim that $\Theta_2(\mathscr{S}_i)\subseteq\mathscr{S}_i$. First note that

$$\Theta_2(E_{ii})=\Phi_1^{-1}(\Theta_1(E_{ii}))=T^{-1}(E_{ii}+S_i)T=E_{ii}+T^{-1}S_iT. \qquad (+)$$

Thus the only nonzero diagonal entry of $\Theta_2(E_{ii})$ is a 1 in position (i,i). If now $E_{kk}\in\mathscr{S}_m$, then

$$\Theta_2(E_{kk})=\zeta_1+\cdots+\zeta_p, \qquad \zeta_j\in\mathscr{S}_j.$$

Since $E_{kk}^2=E_{kk}$ and $\zeta_i\zeta_j=0$ for $i\neq j$, it follows that the ζ_i are orthogonal idempotents, whence from $(+)$, $\Theta_2(E_{kk})=\zeta_k$ and $\zeta_j=0$ for $j\neq k$. Now let $(h,k)\in\Delta_j$, so that $E_{hk}\in\mathscr{S}_m$. Then also

$$\Theta_2(E_{hk})=\delta_1+\cdots+\delta_p, \qquad \delta_j\in\mathscr{S}_j,$$

from which we obtain

$$\Theta_2(E_{hk})=\Theta_2(E_{hk})\Theta_2(E_{kk})=(\delta_1+\cdots+\delta_p)\zeta_k=\delta_k\zeta_k\in\mathscr{S}_m.$$

Thus Θ_2 restricted to any $\mathscr{S}_k$ may be regarded as an automorphism of $\mathscr{T}_{k\,n-1}$. We see that each $\mathscr{T}_{k\,n-1}$ satisfies the hypotheses of the theorem. (Note that the unity of $\mathscr{T}_{k\,n-1}$ is the principal submatrix on $\{2,\ldots,n\}$ of the sum of all E_{jj} with $j\in\Delta_k$.) By the induction hypothesis Θ_2 restricted to $\mathscr{S}_k$ has a factorization

$$\Theta_2|\mathscr{S}_k=\phi_{2k}\circ\bar{\psi}_{2k}\circ\omega_{2k},$$

where ω_{2k} is a permutation automorphism, $\bar{\psi}_{2k}$ is an automorphism of $\mathscr{T}_{k\,n-1}$ induced by an R-automorphism ψ_{2k} of $\mathscr{A}$, and ϕ_{2k} is inner on $\mathscr{T}_{k\,n-1}$ in the sense that there are matrices S_k and $S_k^{(-1)}$ such that $S_kS_k^{(-1)}=S_k^{(-1)}S_k$ is the identity of $\mathscr{T}_{k\,n-1}$ and for any $A\in\mathscr{T}_{k\,n-1}$, $\phi_2(A)=S_k^{(-1)}AS_k$. However, since σ is the identity, then for every k, ω_{2k} is the identity. If we let $S=[1]\oplus(S_1+\cdots+S_p)$, then $S\in T_n$ and is invertible. If we define $\Phi_2(T)=S^{-1}TS$ for all $T\in\mathscr{T}_n$, then Φ_2 extends ϕ_2 to $\mathscr{T}_n$.

Put $\Theta_3=\Phi_2^{-1}\circ\Theta_2$, so that Θ_3 is an R-automorphism of $\mathscr{T}_n$ and $\Theta_3|\mathscr{S}_k=\bar{\psi}_{2k}$. Also $\Theta_3(aE_{11})=\Phi_2^{-1}(\Theta_2(aE_{11}))=S(aE_{11})S^{-1}=aE_{11}$. If $i>1$ and $E_{ij}\in\mathscr{T}_n$, then $E_{ij}\in\mathscr{S}_k$ for some k. Therefore $\Theta_3(aE_{ij})=\psi_{2k}(a)E_{ij}$. Also, if $E_{1j}\in\mathscr{T}_n$, then $\Theta_3(E_{1j})=\Theta_3(E_{11}E_{1j}E_{jj})=E_{11}\Theta_3(E_{1j})E_{jj}$, whence

$\Theta_3(E_{1j}) = b_{1j}E_{1j}$ for some $b_{1j} \in \mathscr{A}$. Furthermore Θ_3^{-1} is an automorphism for which $\Theta_3^{-1}(E_{jj}) = E_{jj}$, $j = 1,\dots,n$, so that $\Theta_3^{-1}(E_{1j}) = c_{1j}E_{1j}$. Thus

$$E_{1j} = \Theta_3\big(\Theta_3^{-1}(E_{1j})\big) = \Theta_3(c_{1j}E_{1j}) = \Theta_3(c_{1j}E_{11}E_{1j})$$
$$= \Theta_3(c_{1j}E_{11})\Theta_3(E_{1j}) = c_{1j}E_{11}b_{1j}E_{1j} = c_{1j}b_{1j}E_{1j}.$$

Therefore $c_{1j}b_{1j} = 1$. Since $E_{1j} = \Theta_3^{-1}(\Theta_3(E_{1j})) = b_{1j}c_{1j}E_{1j}$, we see that b_{1j} and c_{1j} are units. Consequently, for $E_{1j} \in \mathscr{T}_n$ and $j \in \Delta_k$ we have

$$\Theta_3(aE_{1j}) = \Theta_3(aE_{11}E_{1j}) = ab_{1j}E_{1j},$$

as well as

$$\Theta_3(aE_{1j}) = \Theta_3(E_{1j}aE_{jj}) = b_{1j}\psi_{2k}(a)E_{1j}.$$

Since b_{1j} is a unit, we conclude that

$$\psi_{2k}(a) = b_{1j}^{-1}ab_{1j}$$

for all j such that $E_{1j} \in \mathscr{T}_n$ and $j \in \Delta_k$. On the other hand, if $i < j$ and E_{1i}, $E_{1j} \in \mathscr{T}_n$, then by the interpolation property $E_{ij} \in \mathscr{T}_n$. Then

$$b_{1j}E_{1j} = \Theta_3(E_{1j}) = \Theta_3(E_{1i}E_{ij}) = b_{1i}E_{1i}E_{ij} = b_{1i}E_{1j},$$

since $\Theta_3(E_{ij}) = E_{ij}$. Consequently, $b_{1i} = b_{1j}$. Next, since Γ is connected and the Δ_k are the components of Γ^1, then for each k there is a $j_k \in \Delta_k$ such that $E_{1j_k} \in \mathscr{T}_n$. Hence all b_{1j} are equal and all the ψ_{2k} are the same inner automorphism on $\mathscr{A}$. Let $B = 1 \oplus b_{1j}I_{n-1}$, and let $\Phi_3(A) = B^{-1}AB$. Put $\Theta_4 = \Phi_3^{-1} \circ \Theta_3$, so that for all $A \in \mathscr{T}_n$ we obtain

$$\Theta_4(A) = B\Theta_3(A)B^{-1}.$$

We shall show that for all $i \leqslant j$ with $E_{ij} \in \mathscr{T}_n$,

$$\Theta_4(aE_{ij}) = aE_{ij}.$$

First

$$\Theta_4(aE_{11}) = B[\Theta_3(aE_{11})]B^{-1} = aE_{11}.$$

Next, if $E_{1j} \in \mathscr{T}_n$ then

$$\Theta_4(aE_{1j}) = B(ab_{1j}E_{1j})B^{-1} = E_{11}ab_{1j}E_{1j}b_{ij}^{-1}E_{jj} = aE_{1j}.$$

Finally if $i < j$ and $E_{ij} \in \mathscr{T}_n$, we have

$$\Theta_4(aE_{ij}) = B\left[\Theta_3(aE_{ij})\right]B^{-1} = B\left(b_{1j}^{-1}ab_{1j}E_{ij}\right)B = aE_{ij}.$$

Therefore Θ_4 is the identity, and in this case Θ has the desired factorization by the commutation relations (C_1) and (C_2).

Case 2: *σ is not the identity.* Suppose $\sigma(i) = p$ and

$$\Theta_1(E_{ii}) = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & 1 & B \\ 0 & 0 & C \end{bmatrix},$$

where the row and column partitioning is $p-1, 1, n-p$. Since $E_{ii}^k = E_{ii}$ for all k, then $\Theta_1(E_{ii})^k = \Theta_1(E_{11})$ for all k. But since A_1 and C are both strictly upper triangular, so that some power of each is zero, it follows that A_1 and C are themselves zero matrices. Thus

$$\Theta_1(E_{ii}) = \begin{bmatrix} 0 & A_2 & A_3 \\ 0 & 1 & B \\ 0 & 0 & 0 \end{bmatrix}.$$

That is, if $\sigma(i) = p$, then columns $1, \ldots, p-1$ and rows $p+1, \ldots, n$ of $\Theta_1(E_{ii})$ are all zero.

Let P_σ be the permutation matrix corresponding to σ, and let $\Omega(A) = P_\sigma^{-1}AP$. We wish to show that Ω is an automorphism of $\mathscr{T}_n$ so that $\Omega \circ \Theta_1$ satisfies the conditions of case 1. To do this we must show that if (i, j) is an edge of Γ, then so is $(\sigma(i), \sigma(j))$. Given $i < j$, there are three possibilities:

(1) $(\sigma(i), \sigma(j))$ is an edge of Γ;
(2) $\sigma(i) > \sigma(j)$, so that $[i, j]$ is an inversion in σ;
(3) $(\sigma(i), \sigma(j))$ is in $\mathbf{H}$, the complement in $\mathbf{T}$ of Γ.

We shall show that if (i, j) is an edge of Γ, then (2) and (3) cannot hold, so that Γ is invariant under σ.

First suppose that $i < j$, but $\sigma(i) = p > q = \sigma(j)$, so that $[i, j]$ is an inversion. If $E_{ij} \in \mathscr{T}_n$, then

$$\Theta_1(E_{ij}) = \Theta_1(E_{ii})\Theta_1(E_{ij})\Theta_1(E_{jj}).$$

Consider first the product $F = \Theta_1(E_{ii})\Theta_1(E_{ij})$. The rows of F are linear combinations of the rows of $\Theta_1(E_{ij})$ with coefficients from the rows of $\Theta_1(E_{ii})$. But every row of $\Theta_1(E_{ii})$ has zeros in the first $p-1$ positions, whence the rows of F are linear combinations of rows $p, p+1, \dots, n$ of $\Theta_1(E_{ij})$. Since $\Theta_1(E_{ij})$ is upper triangular, we see that all rows of F have zeros in the first $p-1$ positions:

$$F = \begin{bmatrix} 0 & A \\ 0 & T \end{bmatrix},$$

where the partitioning is $p-1, n-p+1$ and where T is upper triangular. Now consider the product

$$F\Theta_1(E_{jj}) = \Theta_1(E_{ij}).$$

Since $\sigma(j) = q$, then rows $q+1, \dots, n$ of $\Theta_1(E_{jj})$ are zero. Also $q+1 \leqslant p$. Again the rows of $F\Theta_1(E_{jj})$ are linear combinations of the rows of $\Theta_1(E_{jj})$ with coefficients from the rows of F. The only rows of $\Theta_1(E_{jj})$ which can have nonzero coefficients are rows $p, p+1, \dots, n$. But since $p \geqslant q+1 = \sigma(j)+1$, all these rows of $\Theta_1(E_{jj})$ are zero. Thus

$$\Theta_1(E_{ij}) = F\Theta_1(E_{jj}) = 0.$$

This is impossible, since Θ_1 is an automorphism. Thus (2) cannot hold.

Next suppose $i < j$ and $\sigma(i) = p < q = \sigma(j)$. We shall show that if (p, q) is an edge of **H**, then so is (i, j). This means that if (i, j) is an edge of Γ, then (3) cannot hold. To the contrary suppose that $E_{ij} \in \mathscr{T}_n$, but $E_{pq} \notin \mathscr{T}_n$. Since every $A \in \mathscr{T}_n$ has a zero in position (p, q), then in particular

$$E_{pp}\Theta_1(E_{ij})E_{qq} = 0.$$

But $E_{pp} = \Theta_1(F_{pp})$ and $E_{qq} = \Theta_1(F_{qq})$ for suitable $F_{pp}, F_{qq} \in \mathscr{T}_n$. So

$$0 = E_{pp}\Theta_1(E_{ij})E_{qq} = \Theta_1(F_{pp}E_{ij}F_{qq}).$$

But Θ^{-1} is an automorphism of $\mathscr{T}_n$, and the corresponding permutation for Θ_1^{-1} is σ^{-1}. Then the argument of the first part of this case applied to Θ_1^{-1} shows that F_{pp} has a 1 in position (i, i) while F_{qq} has a 1 in position (j, j). Then the (i, j) entry of $F_{pp}E_{ij}F_{qq}$ is a 1, whence $E_{pp}E_{ij}E_{qq} \neq 0$, a contradiction, since Θ_1 is an automorphism. Thus if (i, j) is an edge of Γ, so is $(\sigma(i), \sigma(j))$. This completes case 2 and the proof of the theorem. ∎

4. EXAMPLES

Let $C = \Sigma E_{ij}$, where the sum is taken over all i and j such that (i,j) is an edge of Γ. Then C is characteristic for Γ (and for $\mathscr{T}_n$) in that C has a zero in a position iff every $A \in \mathscr{T}_n$ has a zero in that position. Otherwise said, for $i \leqslant j$

$$c_{ij} = \begin{cases} 0, & (i,j) \text{ an edge of } H, \\ 1, & (i,j) \text{ an edge of } \Gamma. \end{cases}$$

Of course $c_{ij} = 0$ for $j < i$. We can specify $\mathscr{T}_n$ and Γ (for given R and $\mathscr{A}$) by giving the matrix C.

EXAMPLE (i). Let

$$C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

This example shows that Γ can be transitive, connected, and interpolatory without being **T**. Thus the result is a proper generalization of the principal theorem of [3].

EXAMPLE (ii). Let

$$C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and let $\mathscr{A} = R$ be the integers. In this case Γ is transitive and connected, but not interpolatory. Define Θ by

$$\Theta(E_{ij}) = \begin{cases} E_{ij}, & (i,j) \neq (1,4), \\ -E_{14}, & (i,j) = (1,4). \end{cases}$$

A direct computation shows that Θ is an automorphism and the Ψ and Ω of the theorem must be the identity. If Θ were inner given by $\Theta(A) = D^{-1}AD$, then since $\Theta(E_{ii}) = E_{ii}$ we have that D is diagonal. But also we have $\Theta(E_{13}) = E_{13}$, $\Theta(E_{23}) = E_{23}$, and $\Theta(E_{24}) = E_{24}$. This implies that $D = dI_4$. But then $D^{-1}(E_{14})D = E_{14}$ while $\Theta(E_{14}) = -E_{14}$. So the interpolatory hypothesis cannot be dropped.

EXAMPLE (iii). Let

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then Γ is transitive and interpolatory (vacuously) but not connected. Let $\mathscr{A}$ be an algebra on which there is a noninner automorphism ψ. For instance we can take $R = \{a + b\sqrt{-5} : a, b \text{ integers}\}$ and $\mathscr{A}$ equal to the 2×2 matrices over R. If

$$A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix},$$

put

$$\Psi(A) = \begin{bmatrix} a_1 & 0 \\ 0 & \psi(a_2) \end{bmatrix}.$$

Then Ψ does not have a factorization as in the conclusion of the theorem. Thus connectedness of Γ cannot be dropped.

REFERENCES

1 G. P. Barker, Automorphism groups of algebras of triangular matrices, *Linear Algebra Appl.* 121:207–215 (1989).
2 G. P. Barker and T. P. Kezlan, The automorphism group of a matrix algebra, in *Current Trends in Matrix Theory* (F. Uhlig and R. G. Grone, Eds.), North-Holland, New York, 1987.
3 G. P. Barker and T. P. Kezlan, Automorphisms of algebras of upper triangular matrices, *Arch. Math.* 55:38–43 (1990).
4 T. P. Kezlan, A note on algebra automorphisms of triangular matrices over commutative rings, *Linear Algebra Appl.* 135:181–184 (1990).

Received 10 October 1989; final manuscript accepted 23 October 1990